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NC branes and hierarchies in quantum Hall fluids

Aziz El Rhalami, El Mostapha Sahraoui and El Hassan Saidi

Lab/UFR, High Energy Physics, Physics Department, Faculty of Science
Av. Ibn Battota, B.P. 1014, Rabat, Morocco
E-mail: arhalami@hotmail.com, sahraoui@theorie.physik.uni-muenchen.de, H-saidi@fsr.ac.ma

ABSTRACT: We develop the non-commutative Chern-Simons gauge model analysis modeling the description of the hierarchical states of fractional quantum Hall fluids. For a generic level n of the hierarchy, we show that the order parameter matrix K_{ab} is given by $\theta^{-2}\,\mathrm{Tr}(\tau_a\tau_b)$, where $\{\tau_a, 1 \leq a \leq n\}$ define a specific set of $n \times n$ matrices depending on the parameter θ and the levels l_a of the CS effective field theory. Our analysis predicts the existence of a third order tensor of order parameters C_{abc} induced by the external magnetic field. It is shown that the C_{abc}'s are not new order parameters and are given by θd_{abc}, where d_{abc} are numbers depending on the l_a's. We also give the generalized quantum Hall soliton extending that obtained in the case of the Laughlin state.

KEYWORDS: Field Theories in Lower Dimensions, D-branes, Superstring Vacua, M(atrix) Theories.

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1. Introduction

Recently it has been proposed that non-commutative (NC) Chern-Simons gauge theory on the $(2+1)$ space may provide a description of the Laughlin state of Fractional Quantum Hall (FQH) fluids [1]–[6]. In this context, it has been shown that the non commutativity parameter θ of the Moyal plane is related to the filling fraction ν_L of the Laughlin state and so to the Chern-Simon effective field coupling λ_{CS} as $\theta B \nu_L = \nu_L \lambda_{\mathrm{CS}} = 1$; with B the external magnetic field and the subscript L stands for Laughlin. In [7], see also [8]–[11], it has been also conjectured that a specific assembly of a system of D0-, D2- and D6-branes and $F1$ strings, stretching between D2- and D6-, has a low energy dynamics similar to the fundamental state of FQH systems. There, the boundary states of the F strings ending on the NC D2-brane are interpreted as the FQH particles. In an external strong magnetic field B, represented by a large number of D0-branes dissolved in D2, the dynamics of these particles is modeled by a non-commutative Chern-Simons (NCCS) U(1) gauge field theory.

In this paper we use these results to develop the NC Chern-Simon theory modeling the description of the hierarchical states of FQH systems. The point is that FQH systems with general expressions of the filling fraction ν are not all of them of Laughlin type [12]; i.e.

with filling fraction $\nu_L = 1/m$, m odd integer. Typical examples are given by states with $\nu(l_1, l_2) = \frac{l_2}{l_1 l_2 - 1}$ where l_1 and l_2 are, respectively, odd and even integers. These kind of states are approached using hierarchical construction ideas [13]–[25]. In the hydrodynamical approach of FQH fluids, the $\frac{l_2}{l_1 l_2 - 1}$ state can be viewed as consisting of two components of incompressible fluids; one describing $\nu_{L,1} = 1/l_1$ FQH state while the other describes the condensation of quasiparticles on the top of the $\nu_{L,1} = 1/l_1$. Put differently, the $\frac{l_2}{l_1 l_2 - 1}$ state can be imagined as a composed system of a Laughlin state of filling fraction $\nu_{L,2} = \frac{1}{l_1(l_2 l_1 - 1)}$ built on an other one with $\nu_{L,1} = 1/l_1$ and satisfying the identity $\nu(l_1, l_2) = \nu_{L,1} + \nu_{L,2}$. As such the total number of particles can, roughly speaking, be thought of as given by the sum $N_1 + N_2$, where N_1 is associated with the state of filling fraction $\nu_{L,1}$ and N_2 with the state $\nu_{L,2}$; see figures 2 and 3 of section 4. These features apply as well for higher orders of the hierarchy with $\nu = \nu(l_1, l_2, \ldots, l_n)$. Current examples correspond to $\nu(3,2) = 2/5 = \frac{1}{3} + \frac{1}{15}$ and $\nu(3,2,2) = 3/7 = \frac{1}{3} + \frac{1}{15} + \frac{1}{35}$.

In the field theory approach, the hierarchical states we refer to here above are described by a $(2+1)$-dimensional system of coupled CS gauge fields whose action reads for generic levels n as, [26, 27, 28]

$$S = \frac{1}{4\pi} \int d^3y \; K_{ab} \partial_\mu A_\nu^a A_\rho^b \epsilon^{\mu\nu\rho} + J_a^\mu A_\mu^a \,. \tag{1.1}$$

In this equation K_{ab} is a $n \times n$ matrix with specific integer entries defining the order parameters characterizing the fluid and carrying the interactions between the CS gauge fields. J_a^μ's are external charge density currents linked for $n = 1$ with the D6-branes charge of the quantum Hall soliton [7, 8]. Note that for this case, the above gauge model reduces to the usual CS effective field theory of the Laughlin state with filling fraction $\nu_L = K_{11}^{-1} = 1/m$. For $n = 2$, however, one deals with a FQH system with filling fraction $\nu(l_1, l_2) = \frac{l_2}{l_1 l_2 - 1}$ and so with hierarchical states of level two. In the Haldane model where $\nu(l_1, l_2)$ may be decomposed as the sum over $\nu_{L,1}$ and $\nu_{L,2}$, the K_{ab} matrix reads then as:

$$K_{ab} = \begin{pmatrix} (2P_1 + 1) & -1 \\ -1 & 2P_2 \end{pmatrix}, \tag{1.2}$$

where we have set $l_1 = 2P_1 + 1$ and $l_2 = 2P_2$. Notice that because of the property $K_{12} \neq 0$; there exists a non-trivial coupling between the two A_μ^1 and A_μ^2 CS gauge fields. This feature will play a crucial role in our present study especially when we build the NCCS theory describing hierarchies by generalizing Susskind method.

Throughout this study we will discuss in details the above mentioned level two Haldane states by developing an adequate generalization of the Susskind construction performed for the Laughlin model. We will also study the case of generic levels of Haldane other hierarchies. Moreover we build the generalization of the quantum Hall soliton and give an interpretation of FQH hierarchies in terms of Dp-branes of uncompactified type-IIA string theory.

The presentation of this paper is as follows. In section 2 we develop the matrix model used to describe level two hierarchical states of FQH liquids with a finite number N of particles. In section 3, we study the NCCS gauge model for the case of Haldane states at level two and consider the infinite limit of N where one-dimensional matrix fields are mapped to $(2+1)$-dimensional fields. In this limit the $\mathrm{U}(N)$ symmetry is replaced by $\mathrm{SDiff}(R^2)$, the group of area preserving diffeomorphisms of the R^2 plane while the matrix commutator becomes a Poisson bracket. In section 4 we build the generalized quantum Hall soliton describing hierarchical states while in section 5 we give results concerning generic values of the level n of the hierarchy. Section 6 is deserved for conclusion.

2. Hierarchy and matrix model for FQH states

In this section we will construct a non-commutative gauge model for the description of hierarchical states of FQH liquids. This model is based on an extension of the Susskind analysis made for the case of the Laughlin state. We first consider the case of a finite number N of electrons; then we aboard the interesting limit when N goes to infinity. The determination of the standard CS effective field theory as the leading term in the NC parameter θ will be worked out explicitly.

To establish the matrix model for the description of FQH hierarchies, we will adopt the following strategy: first we consider a toy model, a matter of introducing some general tools useful for the next steps. Second we develop our matrix model for the description of hierarchical states of level two for a system of $N_1 + N_2 = 2N$ electrons and finally consider the limit N goes to infinity.

2.1 General

To begin consider an electric charged particle, say an electron, moving in the real plane in presence of an external constant magnetic field B. Classically this particle is parametrized by its position $x^i(t)$; $i = 1, 2$ and velocity $v^i = \partial_t x^i$. For a B-field strong enough, the dynamics of the particle is mainly governed by the coupling

$$S[x] = \frac{eB}{2} \int dt \varepsilon_{ij} v^i x^j \,, \tag{2.1}$$

which induces at the quantum level a non commutativity structure on the real plane; i.e. $[x^i, x^j] \propto \epsilon^{ij}/B$. In the case of N classical particles, without mutual interactions, parameterized by the coordinates $x^i_a(t)$ and velocities v^i_a, the dynamics is dominated by the $B - x(t)$ couplings extending eq. (2.1) as $eB\varepsilon_{ij}\Sigma_{a=1}^N v^i_a x^j_a$ and describing a typical strongly correlated system of electrons showing a quantum Hall effect of filling fraction $\nu = N_e/N_\phi$; where $N_\phi = \int_{S^2} B$ and $N_e = N$ are, respectively, the quantum flux and electrons numbers. Quantum mechanically, there are different field theoretical methods to approach the quantum states of this system [20], either by using techniques of non-relativistic quantum mechanics [12], methods of conformal field theory especially for the study of the edge excitations [14, 15] or again by using the CS effective field model [16] describing the limit $N \to \infty$ of electrons. In this case, the CS theory on the $(2+1)$-dimensional space modeling the FQH Laughlin state of filling fraction ν is given by the following action:

$$S[A] = \frac{1}{4\pi\nu} \int d^3y \epsilon^{\mu\nu\rho} \partial_\mu A_\nu A_\rho + \int d^3y J^\mu A_\mu \,, \tag{2.2}$$

The link between this field action and FQH fluids dynamics has been studied in details and most of the results in this direction has been established several years ago [5]–[18]. However an interesting observation has been made recently by Susskind [1] and further considered in [2]–[6] and [8]–[11]. The novelties brought by the study made in [1] is that: (1) Because of the B-field, level n NC Chern-Simons U(1) gauge theory may provide a description of the Laughlin theory at filling fraction $\nu_L = 1/m$. In this vision, eq. (2.2) appears then just as the leading term of a more general theory which reads in general as:

$$S[A] = \frac{1}{4\pi\nu}\int d^3y \epsilon^{\mu\nu\rho}\left[\partial_\mu A_\nu * A_\rho + \frac{2i}{3}A_\mu * A_\nu * A_\rho\right] + \int d^3y J^\mu A_\mu\,, \tag{2.3}$$

where $*$ stands for the usual star operation of non-commutative field theory [29]. (2) The above NCCS U(1) action is in fact the $N \to \infty$ of the following matrix model action:

$$S = \frac{eB}{2}\int dt\left(\epsilon_{ij}\sum_{\alpha=1}^{N}\left[(\dot{X}^i + i[A_0, X^i])X^j + \theta\epsilon^{ij}A_0\right]_{\alpha,\alpha}\right), \tag{2.4}$$

by substituting the $X^i_{\alpha\beta}$'s as:

$$X^i_{\alpha\beta} = y^i\delta_{\alpha\beta} + \theta\varepsilon^{ij}(A_j)_{\alpha\beta}\,, \tag{2.5}$$

$$\left[y^i, y^j\right] = i\theta\varepsilon^{ij}\,. \tag{2.6}$$

At this stage let us give some remarks, they concern some remarkable properties of the above analysis that we will not have the occasion to address in the present paper: (a) The finite matrix model (2.4), which has been conjectured in [17] to describe fractional quantum Hall droplets, was shown to be equivalent to the Calogero integrable model [30] providing then another link between Calogero and Hall systems. (b) The mapping (2.5), which is interpreted as describing fluctuations carried by A_j around the classical solution $X^i = y^i I$, is a kind of background field splitting. It is formally similar to the gauge splitting one uses in the derivation of matrix model from the ten-dimensional Super Yang-Mills theory [31] by using dimensional reduction from $(1+9)$ down to $(1+0)$. Note that in the ideal case where $N = 1$, eq. (2.4) reduces to eq. (2.1) with the constraint eq. (2.6); this ideal situation will be shown later on to be just the leading term of a hierarchical series; see eqs. (2.7) and (4.2).

The analysis we gave here above concerns the Laughlin state; the fundamental state of FQH systems. In what follows we want to generalize it to include hierarchical states. We will start by considering quantum states of level two and too particularly those having a filling fraction $\nu_H(l_1, l_2) = \frac{l_2}{l_1 l_2 - 1}$. Later we turn to the general case.

2.2 1D NC U(2) gauge model

In the purpose of studying FQH states with $\nu_H(l_1, l_2)$, let us first consider a toy model where the $X^i(t) = (X^1, X^2)$ are two one-dimensional 2×2 hermitean matrix fields whose dynamics, in the strong B regime, is described by the following action:

$$S = \frac{eB}{2}\int dt\ \epsilon_{ij}\,\mathrm{Tr}_{\mathrm{u}(2)}\left[\left(\dot{X}^i + i[A_0, X^i]\right)X^j + \theta\epsilon^{ij}A_0\right]. \tag{2.7}$$

In this equation $A_0 = A_0(t)$ is a one-dimensional (1D) gauge field valued in the U(2) algebra; naively it may be thought of as the time component of the U(1) CS gauge field to be considered later on. To fix the idea, the above action may be viewed as associated with the leading term of a general formula; see eq. (3.2).

Since $\mathrm{U}(2) = \mathrm{U}(1)\times\mathrm{SU}(2)$, only the U(1) gauge factor of the gauge field will contribute in the second term of the action (2.7). This action depends linearly on A_0 and so it is just a Lagrange field carrying a field constraint which can be determined by calculating its equation of motion. In doing so, we get the following action of the $X^i(t)$'s fields

$$S = \frac{eB}{2}\int dt\epsilon_{ij}\,\mathrm{Tr}_{\mathrm{u}(2)}(\dot{X}^i X^j)\,, \tag{2.8}$$

together with the 2×2 matrix constraint equation,

$$\left[X^i, X^j\right] = i\theta\epsilon^{ij} I_2\,. \tag{2.9}$$

Expanding the $X^i(t)$ field matrix as follows:

$$X^i = X^i_0 I_2 + Z^i_a\sigma^a\,, \tag{2.10}$$

where I_2 stands for the 2×2 identity matrix and $\sigma^a = (\sigma^x, \sigma^y, \sigma^z)$ are the usual Pauli matrices. Splitting the $X^i_0(t)$ field component, associated with the U(1) factor of U(2), as a sum of constant y^i and a term dependent on time as:

$$X^i_0(t) = (y^i + Z^i_0(t))I_2\,, \tag{2.11}$$

then putting $X^i = y^i I_2 + Z^i$ back into eq. (2.9), we get the following algebra,

$$\left[y^i, y^j\right] = i\theta\epsilon^{ij}\,, \tag{2.12}$$

$$\left[y^i, Z^j\right] = 0\,, \tag{2.13}$$

$$\left[Z^i, Z^j\right] = 0\,. \tag{2.14}$$

Equations (2.13) and (2.14) may be further decomposed using the properties of the Pauli matrices, in particular the Clifford and the su(2) algebraic relations. We find,

$$[y_i, Z_a(t)] = 0\,, \qquad a = 0,1,2,3\,. \tag{2.15}$$

$$\sum_{a=0}^{3}\left[Z^i_a, Z^j_a\right] I_2 + i\sum_{a,b,c=1}^{3}\varepsilon^{abc} Z^i_a Z^j_b\sigma^c = 0\,. \tag{2.16}$$

A natural solution of these equations is obtained by taking $Z_a(t) = 0$, $a = 0,1,2,4$, so that eq. (2.12) describes just the classical solution. Therefore the $Z^i_a(t)$ fields appearing in eqs. (2.10) and (2.11) are interpreted as describing fluctuations around the classical configuration $X^i = y^i I_2$. To get the action describing the $Z^i_a(t)$ fluctuations, we substitute the $X^i(t)$'s by the splitting (2.10) and use eq. (2.12), we get

$$S = \frac{eB}{2}\int dt\epsilon_{ij}\,\mathrm{Tr}_{\mathrm{U}(2)}\left[\left(\dot{Z}^i + i\left[A_0, Z^i\right]\right)Z^j\right]. \tag{2.17}$$

This action is invariant under U(2) automorphisms of the matrix fields, namely

$$\begin{aligned} Z^{i\prime} &= U^{+} Z^{i} U\,, \\ A_0^{\prime} &= U^{+} A_0^{i} U - iU^{+}\frac{\partial}{\partial t} U\,. \end{aligned} \tag{2.18}$$

Equation (2.17) may be generalized by including fermions that we have ignored here above as they are not needed in the present study; it may also be extended by using, instead of 2×2 matrices, higher-dimensional matrix fields. We will consider this situation in section 5 when we consider FQH states with filling fraction $\nu_H(l_1, l_2, \ldots, l_n)$. One of the extensions we are interested in here, which will be used to describe FQH states at level two of hierarchy (SL 2 for short), is based on taking hermitean field matrix valued in the $\mathrm{u}(2)\oplus\mathrm{u}(N)$.

3. NC gauge model for Haldane states

To start consider the system of $N_1 + N_2$ electrons on the real plane parameterized by the coordinates $y^i = (y^1, y^2)$. N_1 should be thought as the number of electrons associated with the underlying Laughlin state of filling fraction $\nu_{L,1} = 1/l_1$. N_2 is a priori the number of particles we get after condensation of quasiparticles [5, 21]; it can be thought of as associated with the filling fraction $\nu_{L,2} = \nu_H(l_1, l_2) - \nu_{L,1} = \frac{1}{l_1(l_1 l_2 - 1)}$. For reasons of simplicity of the formulation of our effective model, we will suppose that $N_1 = N_2 = N$ and consider the case of configurations with a finite number $2N$ of electrons whose coordinates are represented by $2N \times 2N$-dimensional hermitean matrices $X^i(t)$. These are one-dimensional fields valued in the adjoint representation of $\mathrm{U}(2)\oplus\mathrm{U}(N)\subset\mathrm{U}(2N)$. Put differently, the X^i fields have an expansion generalizing eq. (2.10) in the sense that each component Z_a is itself a $N\times N$ hermitean matrix valued adjoint of the group $\mathrm{U}(N)$:

$$Z_a^i = \sum_B T_B Z_a^{i,B}(t)\,, \tag{3.1}$$

where the T_B's are the $\mathrm{U}(N)$ generators. The new matrix model describing the dynamics of SL 2, in presence of a strong B-field, has an action formally similar to eq. (2.4), except now that the $X^i(t)$ and $A_0(t)$ fields are in $Adj_{\mathrm{U}(2)\oplus\mathrm{U}(N)}$ and the trace is taken over the states of the $\mathrm{u}(2)\oplus\mathrm{u}(N)$ algebra. Thus we have,

$$S = \frac{1}{g_2^2}\int dt\epsilon_{ij}\,\mathrm{Tr}_{(\mathrm{u}(2)\oplus\mathrm{u}(N))}\left[(\dot{X} + i[A_0, X^i])X^j + \theta\epsilon^{ij}A_0(1 + J_0)\right], \tag{3.2}$$

where J_0 is the current density of a given external source and where g_2 is a coupling constant to be determined later on; it carries informations on the SL 2 filling fraction ν and the non commutativity parameter θ. The action (3.2) is symmetric under the following change extending eq. (2.18)

$$\begin{aligned} X^{i\prime} &= W^{+} X^{i} W\,, \\ A_0^{\prime} &= W^{+} A_0 W - iW^{+}\frac{\partial}{\partial t} W\,, \end{aligned} \tag{3.3}$$

where $W = U \otimes V$, is a unitary transformation of the $\mathrm{U}(2) \otimes \mathrm{U}(N)$ gauge group. Setting $U = \exp\left(i \sum_{a=1}^{3} \lambda_a \sigma^a\right)$ and $V = \exp(i \sum_{B=1}^{n^2} \Lambda_B T^B)$, the infinitesimal form of the transformations eq. (3.3) reads for the case of $\mathrm{U}(N)$ gauge symmetry for instance as:

$$\begin{aligned} \delta X^i &= -i\left[y^i, \Lambda\right] - i\left[A, X^i\right], \\ \delta A_0 &= \frac{\partial}{\partial t}\Lambda + i\left[\Lambda, A_0\right]. \end{aligned} \tag{3.4}$$

Before going ahead note that due to eq. (2.12), $[y^i, \Lambda]$ behaves as a derivation since,

$$\left[y^i, \Lambda\right] = i\theta\epsilon^{ij}\partial_j\Lambda\,. \tag{3.5}$$

In the limit N goes to infinity the one-dimensional $2N \times 2N$ fields $X^i(t)$ and $A_0(t)$ become infinite matrices; they may be represented by $(2+1)$-dimensional field $X^i(t, y^1, y^2)$ and $A_0(t, y^1, y^2)$ and so is the Z^i fluctuations around the classical solution; all of them are valued in the $\mathrm{U}(2)$ algebra. For later use let us expand this field in terms of the $\mathrm{U}(2)$ generators as,

$$Z^i(y) = Z_0^i(y) I_2 + \sum_{a=1}^{3} Z_a^i(y)\sigma^a\,. \tag{3.6}$$

Moreover when $N \to \infty$ the $\mathrm{U}(2) \otimes \mathrm{U}(N)$ symmetry is also mapped to $\mathrm{U}(2) \times \mathrm{SDiff}(R^2)$; $\mathrm{SDiff}(R^2)$ being the area preserving diffeomorphism group of R^2 plane. So the $Z^i(t, y^1, y^2)$ fluctuations around the classical solution $y^i I$ should be proportional to the space components $A_j(y)$ of the $\mathrm{U}(1)$ Chern-Simons gauge field as

$$Z^i(y) \propto \epsilon^{ij} A_j(y)\,. \tag{3.7}$$

Since the $Z^i(y)$ fluctuations scale as $[Z] = [X] = [y] = L$ while the gauge field A_j scales as L^{-1}, the factor of proportionality should scale like $[X]/[A] = L^2$ as θ does. Here below we study these fluctuations and derive the extension of eq. (2.5) for the case of SL 2.

3.1 Generalized Susskind map

In the decomposition (3.6) involving the dimensionless Pauli matrices, the scaling behaviour of X is completely carried by the Z_a^i component fields. To convert this expansion in term of the A^i gauge fields scaling as $(\text{length})^{-1}$, it is convenient to introduce a new vector basis $\{\tau^1, \tau^2, \tau^3\, \tau^4\}$ of $\mathrm{U}(2)$ related to the standard Pauli matrices basis $\{\sigma^x, \sigma^y, \sigma^z, \sigma^4 \equiv I_2\}$ as:

$$\tau^a = \sum_{b=1}^{4} \Gamma_b^a\; \sigma^b\,; \qquad a, b = 1, 2, 3, 4\,, \tag{3.8}$$

where the entries of the invertible 4×4 matrix Γ scale as $(\text{length})^2$. This change of basis turns out to be very useful when studying the order parameters of FQH fluids; see eqs. (3.14), (3.16), and (3.17). To fix the ideas, let us give hereafter the following special choice for τ_1 and τ_2 in terms of σ^a's,

$$\begin{aligned} \tau_1 &= \left[\alpha_0 I + \alpha_1\sigma^x + \alpha_2\sigma^y\right], \\ \tau_2 &= \left[-\alpha_1\sigma^x - \alpha_2\sigma^y + \alpha_3\sigma^z\right], \end{aligned} \tag{3.9}$$

where we have set $\alpha_1 = \Gamma_1^1 = -\Gamma_1^2$, $\alpha_2 = \Gamma_2^1 = -\Gamma_2^2$, $\alpha_3 = \Gamma_3^2$ and $\alpha_0 = \Gamma_4^4$. Similar formulas may be worked out for τ_3 and τ_4, but we don't need them for the present study. We will show later that the α_a parameters in the above equations are related to the l_1 and l_2 order parameters of the SL 2 configurations and to the non commutativity parameter θ of the plane. Using this τ vector basis, we can expand the gauge fields as

$$A_i(y) = \tau_1 A_i^1(y) + \tau_2 A_i^2(y) + \tau_3 A_i^3(y) + \tau_4 A_i^4(y)\,, \tag{3.10}$$

which upon substituting eq. (3.8), we get the right relation between the Z and A fluctuations.

To get the infinitesimal gauge transformations (3.4), we have to make use of the correspondence rules mapping infinite matrices algebra to the space of functions on the plane. Among them we set:

(i) In the infinite limit, $N \times N$ matrix commutators $[F(t), G(t)]_{\alpha\beta}$ are replaced by the Poisson bracket $\{F(y), G(y)\} = \varepsilon^{kl}\partial_k F \partial_l G$. In other words,

$$\lim_{N\to\infty} i[F(t), G(t)] \longrightarrow \theta\varepsilon^{kl}\frac{\partial F(t,y)}{\partial y^k}\frac{\partial G(t,y)}{\partial y^l}\,.$$

So that the infinitesimal gauge transformation reads as,

$$\delta A_\mu = \partial_\mu \Lambda + \theta\{\Lambda, A_\mu\} + 0(2)\,. \tag{3.11}$$

(ii) the trace $\mathrm{Tr}_{\mathrm{U}(2)\otimes\mathrm{U}(N)}$ operation over the $\mathrm{U}(2)\otimes\mathrm{U}(N)$ adjoint representation states is mapped for $N\to\infty$, to $\int d^2y\, \mathrm{Tr}_{\mathrm{U}(2)}$; $\lim_{N\to\infty}\mathrm{Tr}_{\mathrm{U}(N)}$ should be thought of as $\int d^2y$. Therefore we have, after setting $t = y^0$ and associating the Dirac delta function $\delta^2(y)$ to the $N\times N$ identity matrix I_N, the following correspondence:

$$\lim_{N\to\infty}\frac{1}{N}\int dt\, \mathrm{Tr}_{\mathrm{U}(2)\otimes\mathrm{U}(N)}[\cdots](t) \longrightarrow \int d^3y\, \mathrm{Tr}_{\mathrm{U}(2)}[\cdots](y)\,.$$

Starting from eq. (3.2) and taking $N\to\infty$, the one-dimensional infinite matrix $X(t)$ is mapped to the $(1+2)$ function $X(t,y) = y + A(t,y)$, where $A(t,y)$ is a gauge field valued in the U(2) algebra. Putting back the above expression into eq. (3.2) and following the same analysis made in [1], we get after replacing $\mathrm{Tr}_{\mathrm{U}(N)}\to\int d^2y$, the following NC Chern-Simon gauge theory with a non-abelian U(2) gauge group.

$$S = \frac{1}{4\pi g_2^2}\int d^3y\; \epsilon^{\mu\nu\rho}\,\mathrm{Tr}_{\mathrm{U}(2)}\left[\partial_\mu A_\nu A_\rho - \frac{1}{3}A_\mu\{A_\nu, A_\rho\}\right] + \mathcal{O}(2)\,. \tag{3.12}$$

Like in the Susskind analysis, the $\mathcal{O}(2)$ terms carry higher corrections in the NC parameter which can be obtained by expanding the star product in eq. (2.3). Here we will ignore this detail and so forget about it. The above action is quite similar to eq. (1.1) that we are looking for. A careful inspection shows that eq. (3.12) is not convenient to describe states of level two of the hierarchy as it contains a non-abelian gauge symmetry which is not allowed for the study of FQH hierarchies. In other words the expansion (3.10) and so eq. (3.12) contain too much degrees of freedom, too much more than those appearing in eq. (1.1). They should be then reduced down to two gauge fields only.

Comments. At first sight, the problem of keeping two gauge fields B and $B^{'}$ amongst the four U(2) gauge ones;

$$\mathbf{B} = BY + B^{+}\sigma^{-} + B^{-}\sigma^{+} + B^{'}\sigma^{3} ,$$

Y is the generator of the U(1) factor of U(2), is not a major task in gauge theory with matter. Massless and neutral gauge fields are obtained by breaking U(2) gauge group down to its Cartan subgroup $\mathrm{U}(1)^2$. The two other gauge fields; that is the $B^{\pm}$ fields associated with the U(2) step generators $\sigma^{\pm}$, acquire masses which by an appropriate choice of the Higgs vacuum moduli may be supposed to be heavy enough and then integrated out by eliminating $B^{\pm}$ through their equations of motion; that is: $B^{+} = B^{+}(B, B^{'}, B^{-}; \partial B, \partial B^{'}, \partial B^{-})$ and $B^{-} = B^{-}(B, B^{'}, B^{+}; \partial B, \partial B^{'}, \partial B^{+})$. As a result one gets an effective field theory depending only on the B and $B^{'}$ massless fields. Interactions between B and $B^{'}$, which are generally absent in $\mathrm{U}(1) \times \mathrm{U}(1)$ theories, might come here from the substitution of $B^{+} = B^{+}(B, B^{'}, B^{-}; \partial B, \partial B^{'}, \partial B^{-})$ and $B^{-} = B^{-}(B, B^{'}, B^{+}; \partial B, \partial B^{'}, \partial B^{+})$; but this issue still needs a deeper study. In the brane language, this consists to have a large splitting of the two D2-branes and so too completely separate world volumes.

Though this scenario seems to be a natural way to approach hierarchy, it is however not the unique one can imagine; an other scenario is to think about level two of hierarchy as described by bound states of two D2-branes with gauge fields such as

$$C = U_{11}B + U_{12}B^{'} , \qquad C^{'} = U_{21}B + U_{22}B^{'} ,$$

where B and $B^{'}$ are the gauge fields associated with each of the D2-brane involved in the bound configuration and where U_{ij} are some "similarity" transformations. The reason is that, as far as the effective field theory eqs. (1.1) and (1.2) is concerned, the real symmetric matrix K_{ab} which couples the B and $B^{'}$ fields can be diagonalized leading to the C and $C^{'}$ eigen functions appearing in the above equation. In this scheme, the kinetic terms of the diagonal gauge fields, namely $C\partial C + C^{'}\partial C^{'}$ contain implicitly the B-$B^{'}$ interactions as shown here below:

$$C\partial C + C^{'}\partial C^{'} = U_{11}^2 B\partial B + U_{22}^2 B^{'}\partial B^{'} + U_{12}U_{21}[B\partial B^{'} + B^{'}\partial B] .$$

In this way of doing, interactions between the old gauge fields emerge naturally; but as a counterpart a convincing geometrical interpretation is lacking.

To overcome these difficulties, we shall develop here below a phenomenological method based on imposing adhoc constraints on the fluctuations around the classical solution. We have no rigorous way to derive them, the unique support we can give now is that they lead to the right result once used.

SL 2 **Constraints.** **(C1)** the gauge fluctuations $A_j(y)$ around the classical solution should be carried by two gauge fields $A_j^1(y)$ and $A_j^2(y)$ instead of the four ones involved in the U(2) gauge theory. This means that the U(2) gauge field $A_j(y)$ should be of the form:

$$A_j = \tau_1 A_j^1 + \tau_2 A_j^2 , \tag{3.13}$$

where τ_1 and τ_2 are as in eqs. (3.9).

(C2) In the CS effective model of SL 2 eqs. (1.1) and (1.2), one sees that the above mentioned A^1_μ and A^2_μ gauge fields are coupled to each other through the K_{ab} matrix. Therefore we demand that $\mathrm{tr}(\tau_a\tau_b)$ has moreover a non-diagonal contribution describing the $A^a_\mu - A^b_\mu$ couplings.

$$\begin{aligned} \mathrm{tr}(\tau_a^2) &= \eta_a\,, \\ \mathrm{tr}(\tau_a\tau_b) &\neq 0\,, \qquad \text{for } b \neq a\,. \end{aligned} \tag{3.14}$$

In this equation the η_a parameter is a numerical constant which, for the case of Haldane hierarchy, is equal to l_a/g_2^2.

The model for level 2 states. Putting these two physical constraints back into the action (3.12), we get up to the first order in the non commutativity parameter,

$$S = \frac{1}{4\pi}\int d^3y\,\epsilon^{\mu\nu\rho}\left(K_{ab}\partial_\mu A^a_\nu A^b_\rho + C_{abc}A^a_\mu\{A^b_\nu, A^c_\rho\}\right) + \mathcal{O}(2)\,, \tag{3.15}$$

where $\{A^b_\nu, A^c_\rho\}$ is the usual Poisson bracket and where

$$K_{ab} = \frac{1}{g_2^2}\,\mathrm{Tr}(\tau_a\tau_b)\,, \tag{3.16}$$

$$C_{abc} = \frac{1}{g_2^2}\,\mathrm{Tr}(\tau_a\tau_b\tau_c)\,. \tag{3.17}$$

The leading term of eq. (3.15) is just the usual CS effective field model describing the level two hierarchical states of FQH liquids as shown in eq. (1.1). The second term, however, is the novelty defining a set of order parameters generated by non-commutativity. Actually the action (3.15) can be denoted as S_2; it extends the Susskind action S_1 eq. (2.3) for the first order in θ. Both S_1 and S_2 may be viewed as the two leading terms of a hierarchy of functionals S_n. Moreover given a hierarchy at the level two; that is a 2×2 matrix K_{ab}, one can compute the $\alpha_0, \alpha_1, \alpha_2$ and α_3 parameters involved in eq. (3.9)and then determine the C_{abc} coefficients. To illustrate the method of work, let us first perform the calculations for the level two of the Haldane hierarchy.

3.2 Haldane hierarchy

The order parameters of the level 2 Haldane state of FQH fluids are encoded in the 2×2 matrix K_{ab} given by eq. (1.2). So comparing this equation with eq. (3.16), one can compute explicitly the τ_1 and τ_2 matrices. Straightforward calculations leads to:

$$\begin{aligned} g_2^2 &= 2\bar{\alpha}\alpha\,, \\ g_2^2(2P_1+1) &= 2(\alpha_0^2 + \alpha\bar{\alpha})\,, \\ 2g_2^2P_2 &= 2(\alpha_3^2 + \alpha\bar{\alpha})\,. \end{aligned} \tag{3.18}$$

where P_1 and P_2 are as in eq. (1.2) and where $\alpha = \alpha_1 + i\alpha_2$. As we see these relations define actually links between the parameters of the Haldane SL 2 configuration and the α_a's. To

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better see these relations, let us rewrite eqs. (3.18) into a more convenient form as:

$$\begin{aligned}
2P_1 &= \frac{\alpha_0^2}{|\alpha|^2} \in Z^+ \,, \\
2P_2 &= 1 + \frac{\alpha_3^2}{|\alpha|^2} \in Z^+ \,.
\end{aligned} \tag{3.19}$$

Moreover as the α_a moduli scale as $(\text{length})^2$ exactly like the non commutativity parameter θ of the Moyal plane eq. (2.12), it is then natural to make the following scaling change,

$$\begin{aligned}
\alpha &= \theta\eta \,, \\
\bar{\alpha} &= \theta\bar{\eta} \,,
\end{aligned} \tag{3.20}$$

where now η is a non-zero complex dimensionless number. Note by the way a similar change may be also performed for the α_0 and α_3. However and as we will see hereafter, this feature emerges naturally from the scaling eqs. (3.20). Putting this change back into these relations, we get on one hand

$$g_2 = \theta|\eta|\sqrt{2} \,, \tag{3.21}$$

and on the other hand

$$\alpha_0 = \pm\theta|\eta|\sqrt{2p_1} \,, \tag{3.22}$$

$$\alpha_3 = \pm\theta|\eta|\sqrt{2p_2 - 1} \,. \tag{3.23}$$

Setting $\eta = \eta_1 + i\eta_2$ and grouping altogether the above results, one finds the right fluctuations around the classical solution $X^i = y^i I$ describing the Haldane SL 2:

$$X^i = \begin{pmatrix} X^i_{11} & X^i_{12} \\ X^i_{21} & X^i_{22} \end{pmatrix} , \tag{3.24}$$

where the X^i_{ab}'s are given by:

$$\begin{aligned}
X^i_{11} &= y^i + \theta|\eta|\varepsilon^{ij}\left[(\pm)\sqrt{2p_1}A^1_j + (\pm)\sqrt{(2p_2-1)}A^2_j\right] , \\
X^i_{12} &= \theta\eta\varepsilon^{ij}\left[A^1_j - A^2_j\right] , \\
X^i_{21} &= \theta\bar{\eta}\varepsilon^{ij}[A^1_j - A^2_j] \,, \\
X^i_{22} &= y^i + \theta|\eta|\varepsilon^{ij}\left[(\pm)\sqrt{2p_1}A^1_j - (\pm)\sqrt{(2p_2-1)}A^2_j\right] ,
\end{aligned} \tag{3.25}$$

This is a set of sixteen solutions; they define the generalized Susskind mapping. Moreover, using eqs. (3.17), (3.22) and (3.23); we can also compute the cubic coupling C_{abc}; we find:

$$\begin{aligned}
C_{111} &= \pm\theta|\eta|\,[2P_1 + 3]\sqrt{2P_1} \,, \\
C_{112} &= \pm 2\theta|\eta|\sqrt{2P_1} \,, \\
C_{122} &= \pm 4\theta|\eta|P_2\sqrt{2P_1} \,, \\
C_{222} &= 0 \,.
\end{aligned} \tag{3.26}$$

The remaining other parameters are related to the above ones due to the cyclic property of the trace. Remark that C_{abc} couplings are indeed proportional to θ as expected.

4. Hierarchy and FQH solitons

Following [7], see also [8] a fractional quantum Hall phase similar to the one we have been describing is also observed when studying the low energy dynamics of brane bounds involving D0-, D2- and D6-Branes of the ten-dimensional uncompactified type-IIA superstring. Denoting the IIA string coordinate field variables by $\{t(\tau),\ \varrho(\tau,\sigma),\ \vartheta(\tau,\sigma),\ \varphi(\tau,\sigma),\ \{y^i(\tau,\sigma)\}_{4\leq i\leq 9}\}$, τ and σ are the usual string world sheet variables which should not be confused with the matrices σ^a and τ^a of the U(2) gauge symmetry introduced in previous sections, the above mentioned D-branes bound system, called also quantum Hall soliton, is built for the case of the Laughlin state as follows:

Quantum Hall Soliton. **(a)** A two space dimensional spherical D2-brane parameterized by $\{t, \varrho = R, 0 \leq \vartheta \leq \pi, 0 \leq \varphi \leq 2\pi, \mathbf{0_6}\}$. At fixed time, this D2-Brane is embedded in $R^3 \sim R^+ \times S^2$ and for large values of the radius, D2 may be thought locally of as $R^{1,2}$ which is interpreted as the space time of the CS gauge theory.

(b) N flat six space dimensional D6-branes parameterized by $\{t, \mathbf{0_3}, \{\mathbf{y^i}\}_{\mathbf{4\leq i\leq 9}}\}$ thought of as an external source of charge density $J^0 \propto N\delta^3(x)$ located at the origin $(x^1, x^2, x^3) = (0, 0, 0)$ of the D2-brane.

(c) N fundamental strings F1 stretching between D2 and D6 and parameterized by $\{t, 0 \leq \varrho \leq R, \mathbf{0_2}, \mathbf{0_6}\}$. The string ends on the D2-brane are associated with the electrons of FQH fluids.

(d) MD0-branes dissolved into the D2-brane; They define the flux quanta associated to the external magnetic field B of FQH systems.

In this scheme, the FQH particles in the Laughlin state are described by two $N \times N$ hermitean matrices $X^1(t) \sim R\vartheta(t)$ and $X^2(t) \sim R\varphi(t)$ which can be approximated, for large R, by a flat patch of R^2. In the infinite limit of N and M (strong external magnetic field), the one-dimensional matrix fields are mapped to the (2+1) fields given by $X^i(t, y) = y^i + \theta\varepsilon^{ij}A_j(t, y)$ as discussed in section 3.

For states of the hierarchy of the level two, and more generally for generic n levels with $n \geq 2$, one can also build quantum solitons by extending the above construction. The key point in the generation of hierarchical states out of brane bounds is to suppose that F1 strings end on a collection of nD2-branes in a specific manner. Let us first present the generalized quantum Hall soliton we propose for describing hierarchy; then we make comments.

4.1 Generalized quantum Hall soliton

It is built as follows;[1] see figures 1, 2 and 3:

(a) n coincident spherical D2-brane which we represent as $\{nD2\}$. It has a U(n) symmetry group generated by the euclidean the n^2-dimensional basis $\{\mathbf{t^a}; 1 \leq \mathbf{a} \leq n^2\}$, $\mathbf{t^a}$ being $n \times n$ matrix hermitean matrices satisfying the usual U(n) commutation relations namely $[\mathbf{t^a}, \mathbf{t^b}] = i\, f^{\mathbf{abc}}\, \mathbf{t}^c$ with $f^{\mathbf{abc}}$ the SU(n) completely antisymmetric structure constants. We

[1] If one forgets about constraints on the charged gauge fields we have been using in this paper, the generalised soliton we have given here may be also viewed as describing layers of Quantum Hall Laughlin fluids. The theory one gets then is expected to describe a non-abelian QH states.

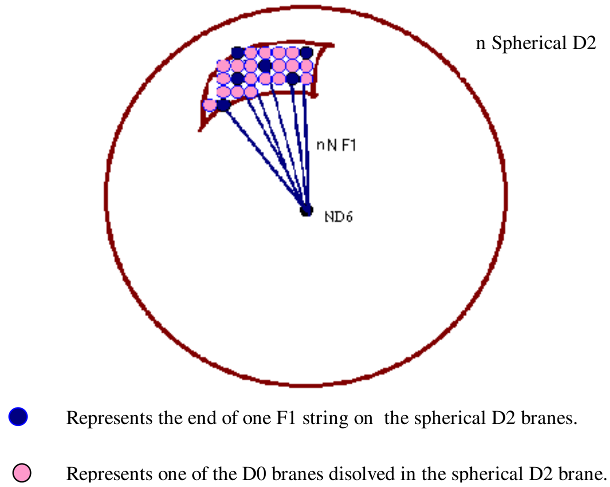

Figure 1: represents a portion of the n coincident spheres describing the generalized fractional Quantum Hall Soliton.

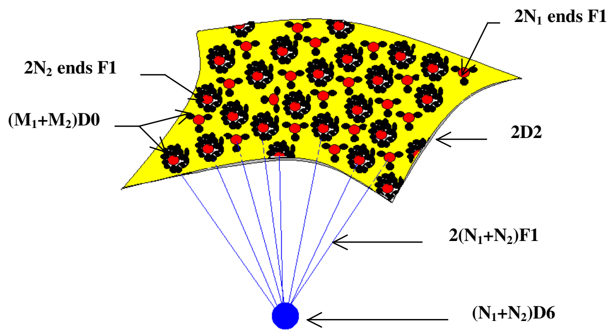

Figure 2: Here is represented the Haldane state of filling fraction $\nu = 2/5$ realized as $\frac{1}{3} + \frac{1}{15}$. States with 3 D0-branes correspond to $\nu_L = 1/3$ and those with 15 D0-branes for $\nu_L = 1/15$.

will also use the SU(n) Cartan-Chevalley basis $\{H_a; E_a^{\pm};\ 1 \leq a \leq n-1\}$; the other SU($n$) generators $E_\alpha^{\pm}$ asociated with non-simple roots α are given by Lie algebra derivations.

(b) A system containing $(N_1 + N_2 + \cdots + N_n)$ flat coincident D6-branes to which we refer to as $\{N_a D6\}_{1 \leq a \leq n}$. They are located at the origin of the three space (X^1, X^2, X^3) and are associated with the different charge densities J_a^0 one encounters in the effective CS gauge model of FQH fluids; eq. (1.1).

(c) A system of $\{nN_1 + nN_2 + \cdots + nN_n\}$ fundamental strings $F1$, $\{nN_a F1\}_{1 \leq a \leq n}$ stretching between the $nD2$ and $\{N_a D6\}_{1 \leq a \leq n}$. The total number $(nN_1 + nN_2 + \cdots + nN_n)$ of $F1$ strings is dictated by the s rule of Hanany-Witten [33] which require, for each D6-brane in the interior of the spherical nD2, $nF1$ strings in order to have a neutral net charge on the D2's.[2]

[2]Hermiticity of the matrix fields is indeed ensured by neutrality of the net charge on the nD2- spherical branes as required by the s-rule of Hanany-Witten. The n^2 hierarchy contribution to $X^i \sim \tau_a \otimes A^j$ is

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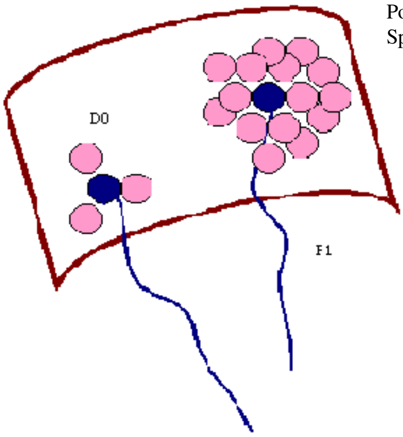

Figure 3: Here we show two F1 string ends of the level two of the hierarchy of filling fraction 2/5 using the splitting $\frac{1}{3}+\frac{1}{15}$. The elementary 1/3(resp. 1/15) state is represented as an F1 string end surrounded by 3 (resp. 15) D0-branes.

String ends on the nD2-branes are associated with the various kinds of particles involved in the building of the hierarchical states of FQH fluids. The various N_a particles are obtained by condensation of quasi-particles which in the present language are associated with the n current densities J_a^0. For simplicity, we suppose that N_a's are all of them equal and so the total number of particles on the D2-branes is n^2N.

Moreover these string sets are not independent; they do interact as required by the effective CS gauge theory for FQH hierarchy. Given two string sets $F1_{(a)}$ and $F1_{(b)}$, their interaction is carried by K_{ab} coupling. In other words if we denote by $X_a^i(t)$ the string end points and $\partial X_a^i/\partial t$ their velocities describing the dynamics of the $\{nN_aF1\}$ subsystem of particles on the D2-branes, then the $X_a^i(t)$-$\frac{\partial X_b^i}{\partial t}$ coupling should be proportional to the hierachical matrix K_{ab} as shown here below,

$$\mathrm{Tr}\left(X^i(t)\frac{\partial X^i}{\partial t}\right)\sim K_{ab}X_a^i(t)\frac{\partial X_b^i}{\partial t}\,. \tag{4.1}$$

(d) MD0-branes, with, $M=(M_1+M_2+\cdots+M_n)$, dissolved in the nD2-branes system. As before, they define the flux quanta associated with the various sets of $F1$ strings; $\{nN_aF1\}_{1\leq a\leq n}$.

4.2 Comments

The generalized quantum Hall soliton describes, amongst others, a set of $(nN_1+nN_2+\cdots+nN_n)$ particles. Upon taking all N_a's equal to N, the system has then n^2N particles and a richer symmetry which make it more or less simple to handle. We will consider hereafter this special case.

carried by $\tau_a\sim adjU(n)$ and turn out to the right factor to fulfill reality of the X^i field.

4.2.1 Matrix model

For finite N_a's, F1 string ends $X^i(t)$ of the full set $\{n^2NF1\} = \bigcup_{a=1}^n\{nN_aF1\}$, with n^2N particles, are valued in $Adj_{\mathrm{U}(n)} \otimes Adj_{\mathrm{U}(N)}$ with very particular coefficients. These coefficients are fixed by the nature of the $F1_{(a)}$ and $F1_{(b)}$ interactions which, for the case of Haldane hierarchy, should be in agreement with the two constraints imposed by the effective CS gauge model of the FQH systems. This is a crucial point in our construction; we propose then to give some explicit details on it. Since the one-dimensional field $X^i(t)$ is in $Adj_{\mathrm{U}(n)} \otimes Adj_{\mathrm{U}(N)}$, let us expand it with respect to the generator basis of $Adj_{\mathrm{U}(n)}$ as follows,

$$X^i(t) = \sum_{\mathbf{a}=1}^{n^2} \mathbf{t}^{\mathbf{a}} X^i_{\mathbf{a}}(t)\,, \tag{4.2}$$

where the $\mathbf{t}^{\mathbf{a}}$ are the $\mathrm{U}(n)$ euclidean generators introduced earlier and where the field components $X^i_{\mathbf{a}}(t)$ are hermitean matrices valued in $Adj_{\mathrm{U}(N)}$. For the special case $n = 1$, describing the fundamental Laughlin state, the above relation is given, around the classical solution $X^i(t) = y^i$, by the Susskind matrix representation

$$\left(X^i(t)\right)_{(n=1)} = y^i + \theta\varepsilon^{ij} A_j\,(t)\,. \tag{4.3}$$

In terms of the components fields, this relation reads as

$$X^i_{\alpha\beta} = y^i\delta_{\alpha\beta} + \theta\varepsilon^{ij}\,(A_j)_{\alpha\beta}\;, \tag{4.4}$$

with $\alpha, \beta = 1, \ldots, n$ and $A_j \sim adj\,\mathrm{U}(N)$. For the hierarchical states described by generic values of n, the analogue of eq. (4.3) satisfying the constraints (4.1) may be easily wondered by using eqs. (2.10) and (2.11) and extending the constraints eqs. (3.13) and (3.14). It reeds as

$$X^i(t) = \sum_{a=1}^{n} X^i_a(t)\,, \tag{4.5}$$

with

$$X^i_a = \frac{1}{n} y^i \mathbf{I}_n + \varepsilon^{ij} A^a_j \tau_a \tag{4.6}$$

and where now the τ_a's are $n \times n$ hermitean matrices of $AdjU(n)$ scaling as $(\text{length})^2$. The A^a_j fields are $N \times N$ hermitean matrix gauge fields of $AdjU(n)$ scaling as $(\text{length})^{-1}$. To have an idea on how these relations may be derived, let us rewrite eq. (4.2) in terms of the $\mathrm{SU}(n)$ Cartan-Chevalley basis $\{H_a; E^{\pm}_a\}_{1\leq a\leq n-1}$. Indeed, the matrix fields $X^i(t)$, describing the string ends on the D2-branes of the full system $\{n^2NF1\}$ becomes

$$\begin{aligned} X^i \;&=\; X^i_0 H_0 + \sum_{a=1}^{n-1}\left(X^i_a H_a + X^{i-}_a E^+_a + X^{i+}_a E^-_a\right) + \\ &+ \sum_{\alpha\ \mathrm{non-simpleroots}} \left(X^{i-}_\alpha E^+_\alpha + X^{i+}_\alpha E^-_\alpha\right), \end{aligned} \tag{4.7}$$

where $H_0 \equiv H_n$ stands for the generator of the $\mathrm{U}(1)$ factor of $\mathrm{U}(n)$, $E^{\pm}_\alpha$ the generators associated with the non-simple roots α of $\mathrm{U}(n)$ and the X^{i-}'s are hermitean conjuagates of X^{i+}. As already discussed in subsection 4.1, the $X^i_{\mathbf{a}}(t)$ are not all of them independent; there are fixed by the $F1$ string couplings of the generalized quantum Hall soliton. It turns out that a class of right constraint equations that lead to the description of the hierarchy, reproducing the result of the fundamental state, are as follows:

$$\begin{aligned}
X^i_0 &= \frac{1}{n} y^i \mathbf{I}_n + \varepsilon^{ij} \sum_{b=1}^{n} \lambda_{nb} A^b_j \,, \\
X^i_a &= \frac{1}{n} y^i \mathbf{I}_n + \varepsilon^{ij} \sum_{b=1}^{n} \lambda_{ab} A^b_j \,, \qquad 1 \leq a \leq n-1 \,, \\
X^{i\pm}_{\mathbf{a}} &= \frac{1}{n} y^i \mathbf{I}_n + \varepsilon^{ij} \sum_{b=1}^{n} \lambda^{\pm}_{\mathbf{a}b} A^b_j \,, \qquad \mathbf{a} = a, \alpha \,.
\end{aligned} \tag{4.8}$$

where the λ's are constants to be compared with the non-commutative geometry parameter θ; they may be determined by solving the constraint equations $\mathrm{Tr}(\tau_a \tau_b) \sim K_{ab}$. The λ's generalize just the one-dimensional non-commutative parameter involved in the analysis on the Laughlin state. Putting these relations back into eqs. (4.7), one finds that the one-dimensional field $X^i(t)$ splits indeed as $X^i(t) = y^i \mathbf{I}_n + \varepsilon^{ij} \sum_{b=1}^{n} A^b_j(t)\, \tau_b$, where the τ_b matrices, carrying the $\{N_a F1\} - \{N_b F1\}$ interactions, are given by,

$$\tau_b = \lambda_{nb} H_n + \sum_{a=1}^{n-1} \left(\lambda_{ab} H_a + \lambda^-_{ab} E^+_a + \lambda^+_{ab} E^-_a \right) + \sum_{\alpha \ \text{non-simpleroots}} \left(\lambda^-_{\alpha b} E^+_\alpha + \lambda^+_{\alpha b} E^-_\alpha \right). \tag{4.9}$$

Later on we will consider a simple example for getting special solutions where only some of the $\lambda_{\mathbf{a}b}$ parameters are non zero and give some explicit details concerning the links between the $\lambda_{\mathbf{a}b}$'s and the physical moduli.

String ends of the $(N_a F1)$ subsystem of $\{(N_a F1)_{1 \leq a \leq n}\}$, are then described by the $N \times N$ hermitean matrix $X^i_a(t)$ eq. (4.6) while the full set of fundamental strings; $\{(N_a F1)_{1 \leq a \leq n}\}$ is described by the system $\{X^i_a(t)\}$; that is by the one-dimensional field $X^i(t)$ valued in $Adj_{\mathrm{U}(n)} \otimes Adj_{\mathrm{U}(N)}$, which we can expand in general as:

$$X^i{}_{\alpha\beta} = y^i \delta_{\alpha\beta} \mathbf{I}_n + \varepsilon^{ij} \sum_{b=1}^{n} \left(A^b_j \right)_{\alpha\beta} \tau_b \,, \tag{4.10}$$

where $\alpha, \beta = 1, \ldots, N$ and where $\mathbf{I}_n$ is the identity $n \times n$ matrix and τ_b are hermitean $n \times n$ matrix given by eqs. (4.9) and scaling as $(\text{length})^2$.

4.2.2 Continous limit

In the limit N infinite, the $F1$ string ends fill all the space of the D2-branes and so the one-dimensional $N \times N$ matrix $(A_j)_{\alpha\beta}(t)$ field is mapped to $(1+2)$-dimensional Chern-Simons gauge field $A_j(t, y^1, y^2)$. Each set of $(nN_a F1)$ strings is then represented by a $(1+2)$-dimensional gauge field A^a_μ and consequently the full $F1$ string ends set $\bigcup_{a=1}^{n} \{(nN_a F1)\}$ is described by the gauge field system $\{A^a_\mu, 1 \leq a \leq n\}$. The one-dimensional matrix model coupling $\mathrm{Tr}(X^i(t) \frac{\partial X^i}{\partial t})$ is mapped to the $(1+2)$ Chern Simons gauge coupling,

$\varepsilon^{ik}\varepsilon^{jl}\sum_{a,b=1}^{n} A_k^a \frac{\partial A_l^b}{\partial t}\operatorname{Tr}(\tau_a\tau_b)$, involving moreover interactions between the various gauge fields in agreement with the effective model eq. (1.1). $F1$ string interactions described by the above CS term are then encoded in the trace $\operatorname{Tr}(\tau_a\tau_b) \sim K_{ab}$. Finally note that the D6-branes appear here as external source of charge which couples to the CS gauge fields.

5. More results

Here we give the results for generic levels of the Haldane hierarchy by following the lines of section 3. In this case FQH hierarchical states at level n are described, for a finite number n^2N of particles, by a one-dimensional $(n\times n)\otimes(N\times N)$ hermitean matrix $X^i(t)$ field valued in the $\mathrm{u}(n)\oplus\mathrm{u}(N)$ algebra. The corresponding matrix model for the strong B regime reads as,

$$S=\frac{1}{g_n^2}\int dt\epsilon_{ij}\operatorname{Tr}_{\mathrm{u}(n)\oplus\mathrm{u}(N)}[(\dot{X}+i[A_0,X^i])X^j+\theta\epsilon^{ij}A_0]\,, \tag{5.1}$$

where g_n is a coupling constant. This action extends the Susskind model as well as the formula (3.2), respectively, obtained by setting $n=1$ and $n=2$. Equation (5.1) defines then a sequence of models in one to one correspondence with FQH hierarchy. For a finite number N, the X^i's may be treated as

$$X^i=[y^i+Z_0^i]\mathbf{I}_n+\sum_{a=1}^{n^2-1}Z_a^i t^a\,, \tag{5.2}$$

where I_n is the $n\times n$ identity matrix, the t^a's are the (n^2-1) $\mathrm{su}(n)$ generators introduced previously and where each component Z_a^i is itself given by a $N\times N$ matrix of $AdjU(N)$. As we have noted in section 3, the Z_a^i fluctuations around the classical configuration are not all of them allowed in the study of FQH hierarchy; only n amongst the n^2 ones are involved in the effective CS model as shown on eq. (1.1). To get the right fluctuations, we shall follow the method we have developped in section 4 by introducing a new vector basis $\{\tau^a;\ 1\le a\le n^2\}$ of the $\mathrm{u}(n)$ algebra. This new basis is related to the old one as

$$\tau^a=\Gamma_b^a\, t^b\,, \tag{5.3}$$

where Γ_b^a is an invertible $n\times n$ matrix. Note that as far as this change is concerned, we will need in practice only n matrices τ^a, $1\le a\le n$ which, without loss of generality, can be taken as:

$$\begin{aligned}
\tau_1 &= [\gamma_1 I+(\delta_1E_1^+ +\bar{\delta}_1E_1^-)]\,,\\
\tau_a &= [-(\beta_aE_{a-1}^+ +\bar{\beta}_aE_{a-1}^-)+\gamma_aH_{a-1}+(\delta_aE_a^+ +\bar{\delta}_aE_a^-)]\,, \qquad 2\le a\le n-1\,,\\
\tau_n &= [-(\beta_nE_{n-1}^+ +\bar{\beta}_nE_{n-1}^-)+\gamma_nH_{n-1}]\,.
\end{aligned} \tag{5.4}$$

In this equation the H_a's and $E_a^\pm$'s are, respectively, the $n\times n$ matrix Cartan generators and Chevalley step operators of the $\mathrm{su}(n)$ algebra while the β_a's, γ_a's and δ_a 's are parameters which should be related to the order parameters of $\mathrm{SL}\,n$. The $E_a^\pm$'s are the generators associated with the α_a simple roots of the su(n) algebra. Notice that the above expression for the τ_a matrices depend on $5n-4$ real moduli; that is n real parameters $\gamma_1, \gamma_2, \ldots, \gamma_n$, $n-1$ complex $\beta_2, \ldots, \beta_n$ and $n-1$ complex $\delta_1, \ldots, \delta_{n-1}$. These moduli are not all of them independent; only a subset of them do. Later on we will show that for the case of Haldane hierarchy there are $n+1$ independent moduli giving the n CS levels l_a and the non commutativity parameter θ.

In the limit N infinite, the 1D u(n)$\oplus$u(N) fields $Z^i(t)$ are mapped to (2+1)-dimensional $Z^i(t, y^1, y^2) \equiv Z^i(y)$ valued in the u(n) algebra which in turns can be set as $Z^i(y) = \varepsilon^{ij} A_j(y)$ as required by the $\mathrm{U}(\infty) \sim \mathrm{SDiff}(R^2)$ invariance. Taking into account all above features and following the same lines we used for the SL 2 mode we get after some algebra,

$$S = \frac{1}{g_n^2} \int d^3y \epsilon^{\mu\nu\rho} \,\mathrm{Tr}_{\mathrm{U}(n)} \left[\partial_\mu A_\nu A_\rho - \frac{i}{3} A_\mu \{A_\nu, A_\rho\}\right] + 0(2)\,. \tag{5.5}$$

Moreover, as we noted in the case of level two of the hierarchy, the expansion (5.5) is not the one needed to describe the FQH hierarchy; it involves n^2 gauge variables while we need n fields only. This means that eq. (5.2) should be constrained as in eq. (4.10).

For the n-th level of the Haldane hierarchy, the SL n constraint equations leading to the appropriate result read as:

$$A_i(y) = \sum_{a=1}^{n} \tau_a A_i^a(y)\,, \tag{5.6}$$

$$\begin{aligned} \mathrm{Tr}(\tau_a \tau_b) - \frac{l_a}{g_n^2}\,\delta_{ab} &\neq 0\,, \qquad \text{for } b = a \pm 1\,, \\ \mathrm{Tr}(\tau_a \tau_b) - \frac{l_a}{g_n^2}\,\delta_{ab} &= 0\,, \qquad \text{otherwise}\,. \end{aligned} \tag{5.7}$$

The τ_a solutions of eqs. (5.7) are indeed given by eqs. (5.4). Putting these constraint equations back into the above action, we get similar relations to those given by eqs. (3.15); but describe now generic n-th levels of the Haldane hierarchy.

$$\begin{aligned} K_{ab} &= \frac{1}{g_n^2}\,\mathrm{Tr}(\tau_a \tau_b)\,, \\ C_{abc} &= \frac{1}{g_n^2}\,\mathrm{Tr}(\tau_a \tau_b \tau_c)\,. \end{aligned} \tag{5.8}$$

These formulas are then valid for any order n of the hierarchy and are, in this sense, universal. Furthermore, using the explicit form of the K matrix of Haldane namely,

$$K_{ab} = l_a \delta_{a,b} - \delta_{a,b-1} - \delta_{a,b+1}\,, \tag{5.9}$$

we can determine the link between the β_a, γ_a and δ_a parameters appearing in eqs. (5.4) and the l_a order parameters. We have,

$$\begin{aligned} g_n^2 &= (\delta_a \bar{\beta}_a + \bar{\delta}_a \beta_a)\,, \\ g_n^2 l_1 &= (n\gamma_1^2 + 2\delta_1 \bar{\delta}_1)\,, \\ g_n^2 l_a &= 2(\gamma_a^2 + \beta_a \bar{\beta}_a + \delta_a \bar{\delta}_a)\,, \\ g_n^2 l_n &= 2(\gamma_n^2 + \beta_n \bar{\beta}_n)\,. \end{aligned} \tag{5.10}$$

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Since Haldane hierarchical states are specified by the l_a levels of the CS gauge model and the θ parameter, the β_a and γ_a moduli should be constrained. A convenient choice for the β_a and γ_a parameters consists of setting set $\beta_a = \delta_a = \beta$, for all values of a. This permits to have the right degrees of freedom one has in Haldane theory; that is: $\gamma_1, \gamma_2, \ldots, \gamma_n$ and β. Therefore the τ_a matrices of eqs. (5.4) are reduced to:

$$\begin{aligned}
\tau_1 &= \gamma_1 I + \beta\,(E_1^+ + E_1^-)\,, \\
\tau_a &= \gamma_a H_{a-1} - \beta\,(E_{a-1}^+ + E_{a-1}^- + E_a^+ + E_a^-)\,, \qquad 2 \le a \le n-1\,, \\
\tau_n &= \gamma_n H_{n-1} - \beta\,(E_{n-1}^+ + E_{n-1}^-)\,.
\end{aligned} \tag{5.11}$$

and so eqs. (5.10) is reduced to

$$\begin{aligned}
g_n^2 &= 2\beta^2\,, \\
g_n^2 l_1 &= (n\gamma_1^2 + 2\beta^2)\,, \\
g_n^2 l_a &= 2(\gamma_a^2 + 2\beta^2)\,, \\
g_n^2 l_n &= 2(\gamma_n^2 + \beta^2)\,.
\end{aligned} \tag{5.12}$$

These equations may be rewritten in the following equivalent form which establishes the link between the CS integers l_1, l_2, l_n and the coupling g_n on one hand and the β and γ_a moduli on the other hand;

$$\begin{aligned}
l_1 &= \left(1 + \frac{n\gamma_1^2}{g_n^2}\right)^{1/2}, \qquad \text{odd integer}\,, \\
l_a &= 2\left(1 + \frac{\gamma_a^2}{g_n^2}\right)^{1/2}, \qquad \text{even integer}\,; \qquad 2 \le a \le n-1\,, \\
l_n &= \left(1 + \frac{2\gamma_n^2}{g_n^2}\right)^{1/2}, \qquad \text{even integer}\,.
\end{aligned} \tag{5.13}$$

Actually the above equations constitute a generalization of the Susskind result on Laughlin state and the level 2 Haldane state we have obtained in section 3, eqs. (3.21). These analysis correspond just the two leading modes SL 1 and SL 2 of a hierarchy of SL n configurations.

6. Conclusion

In this paper we have developed the non-commutative Chern-Simons gauge analysis for the description of the hierarchical states of fractional quantum Hall liquids. For a generic level n of the hierarchy, we have shown that Susskind analysis made for the Laughlin state is naturally generalization for the hierarchical one. Using general features on the CS effective field model of FQH hierarchical states, we have first studied hierarchical states at level two with a special focus on the Haldane hierarchy and then considered the generic case. Among our results:

(a) The derivation of the matrix model describing a set of a finite number n^2N of FQH particles which reads as:

$$S = \frac{1}{g_n^2}\int dt\ \epsilon_{ij}\,\mathrm{Tr}_{\mathrm{u}(n)\oplus\mathrm{u}(N)}[(\dot{X} + i[A_0, X^i])X^j + \theta\epsilon^{ij}A_0]\,, \tag{6.1}$$

where the various quantities appearing in this action were introduced in the core of this paper. Notice that for $n = 1$, this action coincides with that given in [1] and further elaborated in [32] in connection with the study of edge excitations.

(b) The obtention of the generalized mapping, extending the Susskind change $X^i = y^i + \theta\varepsilon^{ij}\ A_j(y)$ made for the Laughlin state, is given by the following $n \times n$ matrix:

$$(X^i_{BD}) = \begin{pmatrix} X^i_{11} & X^i_{12} & \cdots & X^i_{1n} \\ X^i_{21} & X^i_{22} & \cdots & X^i_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ X^i_{n1} & X^i_{n2} & \cdots & X^i_{nn} \end{pmatrix}, \tag{6.2}$$

where

$$X^i_{BD} = y^i \delta_{BD} + \theta\varepsilon^{ij} \left(\sum_{a=1}^{n} A^a_j(y)\ (\tau_a)_{BD} \right), \tag{6.3}$$

and where the τ_a's are as in eqs. (5.11) and (5.12), see also eqs. (4.9). Putting eq. (6.3) back into eq. (6.1), one gets the well-known effective field action (1.1), but also corrections induced by $(2+1)$ space-time non commutativity as shown in eq. (5.5).

(c) the proof that the K_{ab} order parameters are indeed related to the non commutativity θ parameter; they are given by $\theta^{-2}\,\mathrm{Tr}(\tau_a\tau_b)$, where the $\{\tau_a, 1 \le a \le n\}$ set is given by a specific system of $n \times n$ matrices depending on $n+1$ moduli $\gamma_1, \ldots, \gamma_n$ in addition to the coupling constants g_n. The γ_a moduli are shown to be related to the l_a integers of the FQH Chern-Simons effective field theory. These relations were worked out explicitly for the case of Haldane hierarchy as shown in eqs. (5.13).

(d) our analysis predicts moreover the existence of a tensor C_{abc} of induced order parameters. This set of order parameters is shown however to be not a new class as these orders are not really independent. The C_{abc} 's are shown to be given by $\theta^{-2}\,\mathrm{Tr}(\tau_a\tau_b\tau_c) \propto \theta d_{abc}$, where d_{abc} are numbers expressed in terms of the $\mathrm{u}(n)$ Chevalley generators H_a, E^+_a, E^-_a.

Furthermore, we have studied the link between Hierarchical states of FQH fluids and D-branes. By extending the construction of refs [2, 3] associated with the Laughlin state, we have built the generalized quantum Hall soliton supposed to describe generic $\mathrm{SL}\,n$ modes; $n \ge 2$, as a subsystem. As in the case of the Laughlin state, the generalized quantum Hall soliton carries here also much more physics; in particular two coupled CS gauge theories, one describing the electron fluid and the other the fluid of D0-branes. In the present analysis we have considered the special situation where all N_a's are equal. We have supposed that the number N_a of particles one obtains from the a-th condensation is equal to N for any index a. The resulting system has a total number of particles equal to $nN_1 + nN_2 + \cdots + nN_n = n^2N$ and a $\mathrm{U}(n)$ symmetry. It would be interesting to explore the general issue for a system of finite number of particles where the N_a's are different and rebuild the underlying effective non-commutative CS gauge model.

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